**Abstract:** The Lindley distribution is one of the important for studying stress--strength reliability modeling. Besides, some researchers have proposed new classes of distributions based on modifications of the quasi Lindley distribution. In this paper, a new generalized version of this distribution which is called the transmuted quasi Lindley (TQL) distribution is introduced. A comprehensive mathematical treatment of the TL distribution is provided. We derive the $r$th moment and moment generating function this distribution. Moreover, we discuss the least squares, weighted least squares and the maximum likelihood estimation of this distribution.

**Keywords:** Quasi Lindley distribution, Hazard function, Moments, Maximum likelihood estimation.

1. **Introduction and Motivation**

Lindley (1958), introduced a one-parameter distribution, known as Lindley distribution, given by its probability density function. The cumulative distribution function (cdf) and a probability density function (pdf) of the form, respectively:

$$g(x, \theta) = \frac{\theta^2}{1 + \theta} (1 + x)e^{-\theta x}; \quad x > 0, \theta > 0 \quad (1.1)$$

Where $\theta$ is scale parameter. The corresponding cumulative distribution function (cdf) is given by

$$G(x, \theta) = 1 - e^{-\theta x} \left[1 + \frac{\theta x}{1 + \theta}\right] \quad (1.2)$$
Ghitany et al. (2008a) have discussed various properties of this distribution and showed that in many ways Equation (1.1) provides a better model for some applications than the exponential distribution. A discrete version of this distribution has been suggested by Deniz and Ojeda (2011) having its applications in count data related to insurance. Sankaran (1970) obtained the Lindley mixture of Poisson distribution. Ghitany et al. (2008b, c) obtained size-biased and zero-truncated version of Poisson- Lindley distribution and discussed their various properties and applications. Ghitany and Al- mutairi (2009) discussed as various estimation methods for the discrete Poisson- Lindley distribution. Bakouch et al. (2012) obtained an extended Lindley distribution and discussed its various properties and applications. Mazucheli and Achcar (2011) discussed the applications of Lindley distribution to competing risks lifetime data. Ghitany et al. (2011) developed a two-parameter weighted Lindley distribution and discussed its applications to survival data. Zakerzadah and Dolati (2010) obtained a generalized Lindley distribution and discussed its various properties and applications

1.1 Transmutation Map

In this subsection we demonstrate transmuted probability distribution. Let $F_1$ and $F_2$ be the cumulative distribution functions, of two distributions with a common sample space. The general rank transmutation as given in (2007) is defined as

$$G_{R12}(u) = F_2(F_1^{-1}(u)) \quad \text{and} \quad G_{R21}(u) = F_1(F_2^{-1}(u)).$$

Note that the inverse cumulative distribution function also known as quantile function is defined as

$$F^{-1}(y) = \inf_{x \in R} \{F(x) \geq y\} \quad \text{for} \quad y \in [0,1].$$

The functions $G_{R12}(u)$ and $G_{R21}(u)$ both map the unit interval $I = [0,1]$ into itself, and under suitable assumptions are mutual inverses and they satisfy $G_{Rj}(0) = 0$ and $G_{Rj}(0) = 1$. A quadratic Rank Transmutation Map (QRTM) is defined as

$$G_{R12}(u) = u + \lambda u(1-u), |\lambda| \leq 1, \quad (1.3)$$

From which it follows that the cdf's satisfy the relationship

$$F_2(x) = (1+\lambda)F_1(x) - \lambda F_1(x)^2 \quad (1.4)$$

Which on differentiation yields,

$$f_2(x) = f_1(x)[(1+\lambda) - 2\lambda F_1(x)] \quad (1.5)$$

Where $f_1(x)$ and $f_2(x)$ are the corresponding pdfs associated with cdf $F_1(x)$ and $F_2(x)$ respectively. An extensive information about the quadratic rank transmutation map is given in Shaw et al. (2007). Observe that at $\lambda = 0$ we have the distribution of the base random variable. The following Lemma proved that the function $f_2(x)$ in given (1.5) satisfies the property of probability density function.

**Lemma:** $f_2(x)$ given in (1.5) is a well defined probability density function.

**Proof:** Rewriting $f_2(x)$ as $f_2(x) = f_1(x)[(1-\lambda(2F_1(x) - 1)]$ we observe that $f_2(x)$ is nonnegative. We need to show that the integration over the support of the random variable is equal
one. Consider the case when the support of \( f_1(x) \) is \((-\infty, \infty)\). In this case we have

\[
\int_{-\infty}^{\infty} f_2(x)\,dx = \int_{-\infty}^{\infty} f_1(x)[(1 + \lambda) - 2\lambda F_1(x)]\,dx
\]
\[
= (1 + \lambda)\int_{-\infty}^{\infty} f_1(x)\,dx - \lambda\int_{-\infty}^{\infty} 2f_1(x)F_1(x)\,dx
\]
\[
= (1 + \lambda) - \lambda = 1
\]

Similarly, other cases where the support of the random variable is a part of real line follows. Hence \( f_2(x) \) is a well defined probability density function. We call \( f_2(x) \) the transmuted probability density of a random variable with base density \( f_1(x) \). Also note that when \( \lambda = 0 \) then \( f_2(x) = f_1(x) \) This proves the required result.

Many authors dealing with the generalization of some well-known distributions. Aryal and Tsokos (2009) defined the transmuted generalized extreme value distribution and they studied some basic mathematical characteristics of transmuted Gumbel probability distribution and it has been observed that the transmuted Gumbel can be used to model climate data. Also Aryal and Tsokos (2011) presented a new generalization of Weibull distribution called the transmuted Weibull distribution. Recently, Aryal (2013) proposed and studied the various structural properties of the transmuted Log-Logistic distribution, and Muhammad Khan and King (2013) introduced the transmuted modified Weibull distribution which extends recent development on transmuted Weibull distribution by Aryal et al. (2011). and they studied the mathematical properties and maximum likelihood estimation of the unknown parameters.

The rest of the paper is organized as follows. In Section 2 we demonstrate transmuted probability distribution, the hazard rate and reliability functions of TL distribution. In Section 3 we studied the statistical properties include quantile functions, moments, moment generating function. The distribution of order statistics is expressed in Section 4. The least squares and weighted least squares estimators are introduced in Section 5. Finally, In Section 6, we demonstrate the maximum likelihood estimates of the unknown parameters.

2. Transmuted Lindley Distribution

In this section we studied the transmuted Lindley (TL) distribution. Now using (1.1) and (1.2) we have the cdf of transmuted Lindley distribution

\[
F_{TL}(x, \theta, \lambda) = \left[1 - e^{-\theta x} \left(1 + \frac{\theta x}{1 + \theta}\right)\right]\left[1 + \lambda e^{-\theta x} \left(1 + \frac{\theta x}{1 + \theta}\right)\right] \tag{2.1}
\]

where \( \theta \) scale parameter and \( \lambda \) is the transmuted parameter. The restrictions in equation (2.1) on the values of \( \theta \), and \( \lambda \) are always the same. The probability density function (pdf) of the transmuted Lindley distribution is given by
The reliability function \( R_{TL}(x) \) of the transmuted Lindley distribution is denoted by \( R_{TEG}(x) \) also known as the survivor function and is defined as

\[
R_{TL}(x) = 1 - F_{TL}(x) = 1 - \left[ 1 - e^{-\theta x} \left( 1 + \frac{\theta x}{1 + \theta} \right) \right] e^{-\theta x} \left( 1 + \frac{\theta x}{1 + \theta} \right) \]. \tag{2.3}
\]

It is important to note that \( R_{TL}(x) + F_{TL}(x) = 1 \). One of the characteristic in reliability analysis is the hazard rate function (HF) defined by

\[
h_{TL}(x) = \frac{f_{TL}(x)}{1 - F_{TL}(x)} = \frac{\theta^2 (1 + x) e^{-\theta x} \left( 1 - \lambda \right) + 2 \lambda \theta x e^{-\theta x} \left( 1 + \frac{\theta x}{1 + \theta} \right)}{1 - e^{-\theta x} \left( 1 + \frac{\theta x}{1 + \theta} \right) \left[ 1 + \lambda e^{-\theta x} \left( 1 + \frac{\theta x}{1 + \theta} \right) \right]} \]. \tag{2.4}
\]

It is important to note that the units for \( h_{TL}(x) \) is the probability of failure per unit of time, distance or cycles. These failure rates are defined with different choices of parameters. The cumulative hazard function of the transmuted Lindley distribution is denoted by \( H_{TL}(x) \) and is defined as

\[
H_{TL}(x) = -\ln \left[ 1 - e^{-\theta x} \left( 1 + \frac{\theta x}{1 + \theta} \right) \right] \left[ 1 + \lambda e^{-\theta x} \left( 1 + \frac{\theta x}{1 + \theta} \right) \right] \]. \tag{2.5}
\]

It is important to note that the units for \( H_{TL}(x) \) is the cumulative probability of failure per unit of time, distance or cycles. we can show that . For all choice of parameters the distribution has the decreasing patterns of cumulative instantaneous failure rates.

### 3. Statistical Properties

This section is devoted to studying statistical properties of the \( (TL) \) distribution, specifically quantile function, moments and moment generating function.
3.1. Quantile Function

The $q$th quantile $x_q$ of the transmuted Lindley distribution can be obtained from (2.1) as

$$e^{-\theta x}(1 + \frac{\theta x}{1 + \theta}) = \left[ \frac{(\lambda - 1) + \sqrt{\left(\lambda + 1\right)^2 - 4\lambda q}}{2\lambda} \right]$$  \hspace{1cm} (3.1)

We simulate the TL distribution by solving the nonlinear equation

$$e^{-\theta x}(1 + \frac{\theta x}{1 + \theta}) = \left[ \frac{(\lambda - 1) + \sqrt{\left(\lambda + 1\right)^2 - 4\lambda u}}{2\lambda} \right]$$

Where $u$ has the uniform $U(0,1)$ distribution.

3.2. Moments

In this subsection we discuss the $r_{th}$ moment for TL distribution. Moments are necessary and important in any statistical analysis, especially in applications. It can be used to study the most important features and characteristics of a distribution (e.g., tendency, dispersion, skewness and kurtosis).

**Theorem (3.1):** If $X$ has $TL(\theta, \lambda, x)$, then the $r_{th}$ moment of $X$ is given by the following

$$\mu_r(x) = \frac{(1-\lambda)r!(r + \theta + 1)}{\theta'(1 + \theta)} + \frac{2\lambda r!}{(2\theta')(1 + \theta)}$$

$$\times \left[ \frac{(r + 1)(r + 2\theta + 2)}{(1 + \theta)} + r + 2\theta + 1 \right].$$  \hspace{1cm} (3.2)

**Proof:** Let $X$ be a random variable with density function (2.2). The $r_{th}$ ordinary moment of the (TEG) distribution is given by

$$\mu_r(x) = E(X^r) = \int_0^\infty x^r f(x)dx$$

$$= \frac{(1-\lambda)^2}{1 + \theta} \int_0^\infty x^r (1 + x)e^{-\theta x}dx$$

$$+ \frac{2\lambda\theta^2}{1 + \theta} \left\{ \int_0^\infty x^r (1 + x)e^{-2\theta x}dx \right\} + \frac{\theta}{1 + \theta} \int_0^\infty x^{r+1}(1 + x)e^{-2\theta x}dx$$

$$= I + II \hspace{1cm} (3.3)$$
where
\[ I = \int_0^\infty x^r (1 + x) e^{-\theta x} \, dx \]
\[ = \int_0^\infty x^r e^{-\theta x} \, dx + \int_0^\infty x^{r+1} e^{-\theta x} \, dx \]
\[ = \frac{\Gamma(r+1)}{\theta^{r+1}} + \frac{\Gamma(r+2)}{\theta^{r+2}} \] (3.4)

Similarly
\[ II = \int_0^\infty x^r (1 + x) e^{-2\theta x} \, dx + \frac{\theta}{1+\theta} \int_0^\infty x^{r+1} (1 + x) e^{-2\theta x} \, dx \]
\[ = \frac{\Gamma(r+1)}{(2\theta)^{r+1}} + \frac{\Gamma(r+2)}{(2\theta)^{r+2}} + \frac{\theta}{1+\theta} \left[ \frac{\Gamma(r+2)}{(2\theta)^{r+2}} + \frac{\Gamma(r+3)}{(2\theta)^{r+3}} \right] \] (3.5)

Substituting from (3.4) and (3.5) into (3.3) we get
\[ \mu_r(x) = \frac{(1-\lambda)r!(r + \theta + 1)}{\theta' (1 + \theta)} + \frac{2\lambda r!}{(2\theta)' (1 + \theta)} \]
\[ \times \left[ \frac{(r+1)(r+2\theta+2)}{(1+\theta)} + r + 2\theta + 1 \right]. \]

Which completes the proof.

Based on the first four moments of the TL distribution, the measures of skewness \(A(\Phi)\) and kurtosis \(k(\Phi)\) of the TL distribution can obtained as
\[ A(\Phi) = \frac{\mu_3(\theta) - 3\mu_1(\theta)\mu_2(\theta) + 2\mu_1^3(\theta)}{[\mu_2(\theta) - \mu_1^2(\theta)]^2}, \]
(3.6)
and
\[ k(\Phi) = \frac{\mu_4(\theta) - 4\mu_1(\theta)\mu_3(\theta) + 6\mu_1^2(\theta)\mu_2(\theta) - 3\mu_1^4(\theta)}{[\mu_2(\theta) - \mu_1^2(\theta)]^3}. \]
(3.7)

3.3. Moment Generating Function

In this subsection we derived the moment generating function of TL distribution.

**Theorem (3.2):** If \( X \) has TL distribution, then the moment generating function \( M_X(t) \) has the following form
\[ M_X(t) = \frac{\theta^2}{1+\theta} \left( \frac{1-\lambda}{(\theta-t)^2} \right) + \frac{2\lambda(2\theta-t+1)}{(2\theta-t)^3} \left[ \frac{2\theta-t + 1}{1+\theta} \right] \] (3.8)

**Proof:** We start with the well known definition of the moment generating function given by
\[ M_X(t) = E(e^{\alpha t}) = \int e^{\alpha t} f_{TL}(x) \]
\[ = \frac{(1-\lambda)\theta^2}{1+\theta} \int_0^\infty (1+x)e^{-x(\theta-t)}dx \]
\[ + 2\lambda \frac{\theta^2}{1+\theta} \int_0^\infty (1+x)e^{-x(2\theta-t)}(1+\frac{\partial x}{1+\theta})dx \]
\[ = I + II \]  \hspace{1cm} (3.9)

where
\[ I = \int_0^\infty (1+x)e^{-x(\theta-t)}dx = \frac{1}{(\theta-t)} + \frac{1}{(\theta-t)^2}, \]  \hspace{1cm} (3.10)
and
\[ II = \int_0^\infty (1+x)e^{-x(2\theta-t)}(1+\frac{\partial x}{1+\theta})dx \]
\[ = \int_0^\infty (1+x)e^{-x(2\theta-t)}dx + \frac{\theta}{1+\theta} \int_0^\infty (1+x)e^{-x(2\theta-t)}dx \]
\[ = \frac{2\theta-t+1}{(2\theta-t)^2} + \frac{\theta}{1+\theta} \left[ \frac{1}{(2\theta-t)^2} + \frac{1}{(2\theta-t)^3} \right], \]  \hspace{1cm} (3.11)

Substituting from (3.10) and (3.11) into (3.9) we get
\[ M_X(t) = \frac{(1-\lambda)\theta^2}{1+\theta} \left[ \frac{1}{(\theta-t)} + \frac{1}{(\theta-t)^2} \right] \]
\[ + 2\lambda \frac{\theta^2}{1+\theta} \left[ \frac{2\theta-t+1}{(2\theta-t)^2} + \frac{\theta}{1+\theta} \left[ \frac{1}{(2\theta-t)^2} + \frac{1}{(2\theta-t)^3} \right] \right] \]
\[ = \frac{\theta^2}{1+\theta} \left( \frac{(1-\lambda)(\theta-t+1)}{(\theta-t)^2} + \frac{2\lambda(2\theta-t+1)}{(2\theta-t)^3} \left[ 2\theta-t + \frac{1}{1+\theta} \right] \right) \]  \hspace{1cm} (3.12)

Which completes the proof.

4. Distribution of the Order Statistics

In this section, we derive closed form expressions for the pdfs of the \( r_{th} \) order statistic of the TL distribution; also, the measures of skewness and kurtosis of the distribution of the \( r_{th} \) order statistic in a sample of size \( n \) for different choices of \( n, r \) are presented in this section. Let \( X_1, X_2, \ldots, X_n \) be a simple random sample from TL distribution with pdf and cdf given by (2.1) and (2.2), respectively.
Let \( X_1, X_2, \ldots, X_n \) denote the order statistics obtained from this sample. We now give the probability density function of \( X_{r:n} \), say \( f_{r:n}(x, \theta, \lambda) \) and the moments of \( X_{r:n}, r = 1, 2, \ldots, n \). Therefore, the measures of skewness and kurtosis of the distribution of the \( X_{r:n} \) are presented. The probability density function of \( X_{r:n} \) is given by

\[
f_{r:n}(x, \theta, \lambda) = \frac{1}{B(r, n-r+1)} \left[ F(x, \theta, \lambda) \right]^{-1} \left[ 1 - F(x, \theta, \lambda) \right]^{r-1} f(x, \theta, \lambda) \tag{4.1}
\]

where \( F(x, \theta, \lambda) \) and \( f(x, \theta, \lambda) \) are the cdf and pdf of the TL distribution given by (2.1), (2.2), respectively, and \( B(\cdot, \cdot) \) is the beta function, since \( 0 < F(x, \theta, \lambda) < 1 \), for \( x > 0 \), by using the binomial series expansion of \( \left[ 1 - F(x, \theta, \lambda) \right]^{r-1} \), given by

\[
\left[ 1 - F(x, \theta, \lambda) \right]^{r-1} = \sum_{j=0}^{n-r} (-1)^j \binom{n-r}{j} [F(x, \theta, \lambda)]^j. \tag{4.2}
\]

We have

\[
f_{r:n}(x, \theta, \lambda) = \frac{1}{B(r, n-r+1)} \sum_{j=0}^{n-r} (-1)^j \binom{n-r}{j} [F(x, \theta, \lambda)]^j f(x, \theta, \lambda), \tag{4.3}
\]

Substituting from (2.1) and (2.2) into (4.3), we can express the \( k_{th} \) ordinary moment of the \( r_{th} \) order statistics \( X_{r:n} \) say \( E(X_{r:n}^k) \) as a linear combination of the \( k_{th} \) moments of the TL distribution with different shape parameters. Therefore, the measures of skewness and kurtosis of the distribution of \( X_{r:n} \) can be calculated.

5. Least Squares and Weighted Least Squares Estimators

In this section we provide the regression based method estimators of the unknown parameters of the transmuted Lindley distribution which was originally suggested by Swain, Venkatraman and Wilson (1988) to estimate the parameters of beta distributions. It can be used some other cases also. Suppose \( Y_1, \ldots, Y_n \) is a random sample of size \( n \) from a distribution function \( G(.) \) and suppose \( Y_{(i)} ; i = 1, 2, \ldots, n \) denotes the ordered sample. The proposed method uses the distribution of \( G(Y_{(i)}) \). For a sample of size \( n \), we have

\[
E(G(Y_{(j)})) = \frac{j}{n+1}, \quad V(G(Y_{(j)})) = \frac{j(n-j+1)}{(n+1)^2(n+2)}
\]

and \( \text{Cov}(G(Y_{(j)}), G(Y_{(k)})) = \frac{j(n-k+1)}{(n+1)^2(n+2)} \); for \( j < k \),

see Johnson, Kotz and Balakrishnan (1995). Using the expectations and the variances, two variants of the least squares methods can be used.
Method 1 (Least Squares Estimators)

Obtain the estimators by minimizing

$$
\sum_{j=1}^{n} \left( G(Y_{(j)} - \frac{j}{n+1}) \right)^2,
$$

(5.1)

with respect to the unknown parameters. Therefore in case of TL distribution the least squares estimators of $\theta$ and $\lambda$, say $\hat{\theta}_{LSE}$ and $\hat{\lambda}_{LSE}$ respectively, can be obtained by minimizing

$$
\sum_{j=1}^{n} \left[ 1 - e^{-\theta_{(j)}} \left( 1 + \frac{\hat{k}_{(j)}}{1 + \theta} \right) \right] \left[ 1 + \lambda e^{-\theta_{(j)}} \left( 1 + \frac{\hat{k}_{(j)}}{1 + \theta} \right) \right] - \frac{j}{n+1} \right]^2
$$

with respect to $\theta$, and $\lambda$.

Method 2 (Weighted Least Squares Estimators)

The weighted least squares estimators can be obtained by minimizing

$$
\sum_{j=1}^{n} w_j \left( G(Y_{(j)} - \frac{j}{n+1}) \right)^2,
$$

(5.2)

with respect to the unknown parameters, where

$$
w_j = \frac{1}{V(G(Y_{(j)}))} = \frac{(n+1)^2(n+2)}{j(n-j+1)}.
$$

Therefore, in case of TL distribution the weighted least squares estimators of $\theta$ and $\lambda$, say $\hat{\theta}_{WLS}$ and $\hat{\lambda}_{WLS}$ respectively, can be obtained by minimizing

$$
\sum_{j=1}^{n} w_j \left[ 1 - e^{-\theta_{(j)}} \left( 1 + \frac{\hat{k}_{(j)}}{1 + \theta} \right) \right] \left[ 1 + \lambda e^{-\theta_{(j)}} \left( 1 + \frac{\hat{k}_{(j)}}{1 + \theta} \right) \right] - \frac{j}{n+1} \right]^2
$$

with respect to the unknown parameters only.

6. Estimation and Inference

In this section, we determine the maximum likelihood estimates (MLEs) of the parameters of the TL distribution from complete samples only. Let $X_1, X_2, ..., X_n$ be a random sample of size $n$ from $TL(\theta, \lambda, x)$. The likelihood function for the vector of parameters $\Phi = (\theta, \lambda)$ can be written as
\[ Lf(x_{(i)}, \Phi) = \Pi_{i=1}^{n} f(x_{(i)}, \Phi) \]
\[ = \left( \frac{\theta^2}{1 + \theta} \right)^n \Pi_{i=1}^{n} (1 + x_{(i)}) e^{-\frac{\theta}{1+\theta} x_{(i)}} \]
\[ \times \Pi_{i=1}^{n} \left\{ 1 - \lambda + 2\lambda e^{-\theta x_{(i)}} (1 + \frac{\theta x_{(i)}}{1+\theta}) \right\}. \quad (6.1) \]

Taking the log-likelihood function for the vector of parameters \( \Phi = (\theta, \lambda) \) we get
\[ \log L = 2n \log \theta - n \log(1 + \theta) - \theta \sum_{i=1}^{n} x_{(i)} + \sum_{i=1}^{n} \log(1 + x_{(i)}) \]
\[ + \sum_{i=1}^{n} \log \left\{ 1 - \lambda + 2\lambda e^{-\theta x_{(i)}} (1 + \frac{\theta x_{(i)}}{1+\theta}) \right\}. \quad (6.2) \]

The log-likelihood can be maximized either directly or by solving the nonlinear likelihood equations obtained by differentiating (6.2). The components of the score vector are given by
\[ \frac{\partial \log L}{\partial \theta} = \frac{2n}{\theta} - \frac{n}{1+\theta} - \sum_{i=1}^{n} x_{(i)} \]
\[ + \sum_{i=1}^{n} \left\{ \frac{2\lambda x_{(i)} e^{-\theta x_{(i)}} (\frac{1}{1+\theta})^2 - 2x_{(i)} \lambda e^{-\theta x_{(i)}} (1 + \frac{\theta x_{(i)}}{1+\theta})}{1 - \lambda + 2\lambda e^{-\theta x_{(i)}} (1 + \frac{\theta x_{(i)}}{1+\theta})} \right\} = 0, \quad (6.3) \]

and
\[ \frac{\partial \log L}{\partial \lambda} = \sum_{i=1}^{n} \left\{ \frac{2e^{-\theta x_{(i)}} (1 + \frac{\theta x_{(i)}}{1+\theta}) - 1}{1 - \lambda + 2\lambda e^{-\theta x_{(i)}} (1 + \frac{\theta x_{(i)}}{1+\theta})} \right\} = 0. \quad (6.4) \]

We can find the estimates of the unknown parameters by maximum likelihood method by setting these above non-linear equations (6.3) and (6.4) to zero and solve them simultaneously. Therefore, we have to use mathematical package to get the MLE of the unknown parameters.

References


