Application of the Leray-Schauder Alternative for a Certain Class of Non Linear Singular Integral Equations

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Abstract: In this study, we study the problem of existence of solution of nonlinear singular integral equations of the form

\[ w(z) = f_1(z, w(z), h(z), T_0 g_1(., w(\cdot), h(\cdot))) (z) \]
\[ h(z) = f_2(z, w(z), h(z), \Pi_c g_2(., w(\cdot), h(\cdot))) (z) \]

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1. Introduction:

Let \( G \subset C \) be a simply connected region with smooth boundary. As known, the system of real partial differential equations of the form

\[ u_x - v_y = H_1(x, y, u, v, u_x, v_x, u_y, v_y) \]
\[ u_y + v_x = H_2(x, y, u, v, u_x, v_x, u_y, v_y) \]

is equivalent to the complex partial differential equation

(1.1) \[ \partial_z w = F(z, w, \partial_z w) \]

where

\[ w = u + iv, z = x + iy, \partial_z = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right), \partial_{\bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \]

The existence of the solution of the equation (1.1) satisfying the Dirichlet boundary conditions
Gzczw, GCgzgw, G∈=∂∈=∂000, Im(α), (Re(α), in Holder space (GCα), under suitable conditions, had been investigated by Ishak Altun and Kerim Koca (see[2]).

Let the function F in (1.1) be a complex valued scalar function defined on the region

\[ D = \{(z,w,h) : z \in \overline{G}, w, h \in C\} = \overline{G} \times C^2 \]

and let us consider the operators

\begin{align*}
T_G f(z) &= -\frac{1}{\pi} \int_{\partial G} \frac{f(\zeta)}{\bar{\zeta} - z} d\bar{\xi}d\eta, \\
\Pi_G f(z) &= -\frac{1}{\pi} \int_{\partial G} \frac{f(\zeta)}{(\zeta - z)^2} d\bar{\xi}d\eta,
\end{align*}

for \( f \in C^a(\overline{G}) \). In this case, the solutions \( w \) of the equation (1.1) satisfy the system of nonlinear singular integral equations

\begin{align*}
w(z) &= \phi(z) + T_G F(. , w(), h())(z) \\
h(z) &= \phi(z) + \Pi_G F(. , w(), h())(z)
\end{align*}

where \( h = \partial_\xi w \) and \( \phi(z) \)'s are arbitrary holomorphic functions defined on \( G \). In [4], we studied, the more general nonlinear singular integral system

\begin{align*}
w(z) &= f_1(z, w(z), h(z), T_G g_1(. , w(), h())(z)) \\
h(z) &= f_2(z, w(z), h(z), \Pi_G g_2(. , w(), h())(z)
\end{align*}

In this paper we continue to study the (1.2) system using Leray Schauder alternative.

2. Results and Discussion:

The Existence, Uniqueness and Determination of the Solution the System of Singular Integral Equations.

In this section, we will present some theorems related to the solutions of the system (1.2) under suitable conditions.

**Definition 1.** If for every \( z_1, z_2 \in \overline{G} \) there are constants \( H > 0 \) and \( \alpha \) satisfying the inequality

\[ |w(z_2) - w(z_1)| \leq H|z_2 - z_1|^{\alpha}, 0 < \alpha < 1 \]

then the function \( w : \overline{G} \to C \) is said to satisfy the Holder condition in the region \( \overline{G} \) or to be Holder continuous. Let us denote the class of Holder continuous functions defined on \( \overline{G} \) by \( C^a(\overline{G}) \). This class is a vector space. On the other hand, \( C^{(0)}(\overline{G}) \equiv C(\overline{G}) \) is the class of the continuous functions on \( \overline{G} \) and for \( w \in C(\overline{G}) \) in this class, the norm is defined to be

\[ \|w\|_{\overline{G}} = \sup_{z \in \overline{G}} |w(z)|. \]
On the other hand, if the norm for \( w \in C^\alpha(G) \) is defined as
\[
\|w\|_{\alpha} = \left\| w \right\|_{c^\alpha(G)} = \|w\|_{\infty} + H(w, \alpha)
\]
where
\[
H(w, \alpha) = \sup \left\{ \left| w(z_1) - w(z_2) \right| z_1 - z_2^{-\alpha} : z_1 \neq z_2, z_1, z_2 \in \overline{G} \right\}
\]
then the class \( C^\alpha(G) \) becomes a Banach space with this norm.

Let us denote the Holder continuous functions defined on \( \overline{G} \) and having partial derivatives of first order with respect to variables \( z \) and \( \bar{z} \) by \( C^{(1, \alpha)}(G) \). This class constitutes a Banach space with norm
\[
\|w\|_{1,\alpha} = \left\| w \right\|_{c^{(1, \alpha)}(G)} = \max \left\{ \left\| w \right\|_{\alpha}, \left\| \partial_z w \right\|_{\alpha}, \left\| \partial_{\bar{z}} w \right\|_{\alpha} \right\}
\]
for \( w \in C^{(1, \alpha)}(G) \).

Moreover the vector spaces
\[
C^2(G) = C(G) \times C(G) = \left\{ (w, h) : w, h \in C(G) \right\}
\]
\[
C^{(2, \alpha)}(G) = C^{(1, \alpha)}(G) \times C^{(1, \alpha)}(G) = \left\{ (w, h) : w, h \in C^{(1, \alpha)}(G) \right\}
\]
having norms.
\[
\left\| (w, h) \right\|_{2, \alpha} = \left\| (w, h) \right\|_{c^{(2, \alpha)}(G)} = \max \left\{ \|w\|_{\infty}, \|h\|_{\infty} \right\}
\]
\[
\left\| (w, h) \right\|_{\alpha, 2} = \left\| (w, h) \right\|_{c^{(2, \alpha)}(G)} = \max \left\{ \|w\|_{\alpha}, \|h\|_{\alpha} \right\}
\]
become Banach spaces. We denote these spaces by
\[
\left( C^2(G) \right)_{\left\| \cdot \right\|_{2, \alpha}}, \text{ ve } \left( C^{(2, \alpha)}(G) \right)_{\left\| \cdot \right\|_{\alpha, 2}}
\]
respectively.

**Definition 2.** Let \( h : \overline{D} \rightarrow C \) where \( \overline{D} = G \times C^3 \) be given. If for every \((z_1, p_1, q_1, r_1), (z_2, p_2, q_2, r_2) \in \overline{D} \) there are positive numbers \( l_1, l_2, l_3, l_4 \) satisfying
\[
| h(z_1, p_1, q_1, r_1) - h(z_2, p_2, q_2, r_2) | \leq l_1 | z_1 - z_2 |^\alpha + l_2 | p_1 - p_2 |
+ l_3 | q_1 - q_2 | + l_4 | r_1 - r_2 |
\]
then the function \( h \) is said to be of class \( H_{\alpha,1,1,1} \left( l_1, l_2, l_3, l_4; \overline{D} \right) \) over \( \overline{D} \) and we write \( h \in H_{\alpha,1,1,1} \left( l_1, l_2, l_3, l_4; \overline{D} \right) \).

**Definition 3.** Let \( h^* : \overline{D}_1 \rightarrow C \) where \( \overline{D}_1 = G \times C^2 \). If for every \((z_1, p_1, q_1), (z_2, p_2, q_2) \in \overline{D}_1 \) there are positive numbers \( m_1, m_2, m_3 \) satisfying
\[
|h^*(z_1, p_1, q_1) - h^*(z_2, p_2, q_2)| \leq m_1 | z_1 - z_2 |^\alpha + m_2 | p_1 - p_2 |
+ m_3 | q_1 - q_2 |
\]

\[
| h(z_1, p_1, q_1) - h(z_2, p_2, q_2) | \leq m_1 | z_1 - z_2 |^\alpha + m_2 | p_1 - p_2 |
+ m_3 | q_1 - q_2 |
\]

\[
\left\| (w, h) \right\|_{1,\alpha} = \max \left\{ \left\| w \right\|_{\alpha}, \left\| h \right\|_{\alpha} \right\}
\]
for \( w \in C^{(1, \alpha)}(G) \).

Moreover the vector spaces
\[
C^2(G) = C(G) \times C(G) = \left\{ (w, h) : w, h \in C(G) \right\}
\]
\[
C^{(2, \alpha)}(G) = C^{(1, \alpha)}(G) \times C^{(1, \alpha)}(G) = \left\{ (w, h) : w, h \in C^{(1, \alpha)}(G) \right\}
\]
having norms.
\[
\left\| (w, h) \right\|_{2, \alpha} = \max \left\{ \|w\|_{\infty}, \|h\|_{\infty} \right\}
\]
\[
\left\| (w, h) \right\|_{\alpha, 2} = \max \left\{ \|w\|_{\alpha}, \|h\|_{\alpha} \right\}
\]
become Banach spaces. We denote these spaces by
\[
\left( C^2(G) \right)_{\left\| \cdot \right\|_{2, \alpha}}, \text{ ve } \left( C^{(2, \alpha)}(G) \right)_{\left\| \cdot \right\|_{\alpha, 2}}
\]
respectively.
then the function \( h^* \) is said to be of class \( H_{\alpha,1,1} \left( m_1, m_2, m_3, l_4 : \overline{D_1} \right) \) over \( \overline{D_1} \) and we write \( h \in H_{\alpha,1,1} \left( m_1, m_2, m_3, l_4 : \overline{D_1} \right) \).

Let for the bounded operators

\[
\begin{align*}
\| T_G \|_\alpha & = \sup \left\{ \| T_G w \|_\alpha : w \in C^{(\alpha)}(\overline{G}) \|w\|_\alpha < 1 \right\} \\
\| \Pi_G \|_\alpha & = \sup \left\{ \| \Pi_G w \|_\alpha : w \in C^{(\alpha)}(\overline{G}) \|w\|_\alpha < 1 \right\}
\end{align*}
\]

**Lemma 1[4]:** Let \( f_k \in H_{\alpha,1,1} \left( l_{k_1}, l_{k_2}, l_{k_3}, l_{k_4} : \overline{D} \right) \), \( g_k \in H_{\alpha,1,1} \left( m_{k_1}, m_{k_2}, m_{k_3}, l_4 : \overline{D} \right) \) \((k = 1, 2)\)

\[
\theta = (0,0) \quad \text{and} \quad S_\alpha (\theta, R) = \left\{ (w, h) : \| (w, h) \|_{\alpha,2} \leq R \right\}
\]

If

\[
\begin{align*}
l_{0k} & = \max \left\{ f_k (0,0,0,0) : z \in \overline{G} \right\} \\
m_{ak} & = \max \left\{ g_k (0,0) : z \in \overline{G} \right\}
\end{align*}
\]

\[
\begin{align*}
K_1 & = l_{01} + l_{11} + 2(l_{12} + l_{13})R + 2l_{14} [m_{01} + 2m_{11} + 2(m_{12} + m_{13})] R \| T_G \|_\alpha \\
K_2 & = l_{02} + l_{21} + 2(l_{22} + l_{23}) R + 2l_{24} [m_{02} + 2m_{21} + 2(m_{22} + m_{23})] R \| \Pi_G \|_\alpha
\end{align*}
\]

and then for

\[
\begin{align*}
\tilde{w}(z) & = f_1 (z, w(z), h(z), T_G g_1 (., w(\cdot), h(\cdot))(z)) \\
\tilde{h}(z) & = f_2 (z, w(z), h(z), \Pi_G g_2 (., w(\cdot), h(\cdot))(z))
\end{align*}
\]

the operator

\[
A : C^{(\alpha,2)}(\overline{G}) \rightarrow C^{(\alpha,2)}(\overline{G}) \quad 0 < \alpha < 1
\]

\[
(w, h) \rightarrow A(w, h) = (\tilde{w}, \tilde{h})
\]

transforms the ball \( S_\alpha (\theta; R) \) into the ball \( S_\alpha (\theta; \max\{K_1, K_2\}) \) itself.

**Theorem 1[5]**

Let \( E \) be a Banach space, \( C \) a closed, convex subset of \( E \), \( U \) an open subset of \( C \) and \( 0 \in U \).

Suppose that \( A : \overline{U} \rightarrow C \) is a continuous, compact (that is, \( A(\overline{U}) \) is a relative compact subset of \( C \)) map. Then either

(A1) \( A \) has a fixed point in \( \overline{U} \), or

(A2) there exist \( u \in \partial U \) (the boundary of \( U \) in \( C \)) and \( \lambda \in (0,1) \) with \( u = \lambda Au \)

Our main result is given in the following theorem:

**Theorem 2**

Let \( \theta = 0 \in C_1^\alpha(\overline{G}) \) \( 0 < \alpha < 1 \) and \( f_1, f_2 \in H_{\alpha,1,1} \left( l_1, l_2, l_3, l_4 : \overline{D} \right) \). Suppose that the condition
$R \leq \max\{K_1, K_2\}$ holds. In addition suppose that for any solution $(w, h) \in C^{a,2}(\bar{G})$ of the system

$$
\begin{align*}
   w(z) &= A_f(z, w(z), h(z), T_g g_1 (\cdot, w(\cdot), h(\cdot))(z)) \\
   h(z) &= A'_f(z, w(z), h(z), T_g g_2 (\cdot, w(\cdot), h(\cdot))(z))
\end{align*}
$$

for each $\lambda \in (0,1)$ we have $\|(w, h)\|_{a,2} \neq R$. Then the system (1.2) has a solution $(w, h) \in S_a(\theta; R)$.

**Proof:** To prove the existence of solution (1.2) we apply Theorem 1. We consider the space $\left(\bigcap_{a,2}^{a,2}(\bar{G})\right)$ as $E$, the set $S_a(\theta; \max\{K_1, K_2\})$ as $C$ and the set int $B^{a,2}(\theta; R)$ (the interior of $S^{a,2}(\theta; R)$ ) as $U$ in Theorem 1. It is obvious that the subset $S_a(\theta; \max\{K_1, K_2\})$ is closed and convex. $\theta = (0,0) \in S_a(\theta; R)$. Now we consider the operator $A$ defined as in Lemma 2. $A$ transforms $S_a(\theta; R)$ to $S_a(\theta; \max\{K_1, K_2\})$. $A$ is continuous [3],[4]. Now we shall prove that $A$ is compact. For $(w, h) \in S^{a,2}(\theta; \max\{K_1, K_2\})$ we have $\|(w, h)\|_{a,2} \leq R$. Therefore, if for any $\varepsilon > 0$ we choose $\delta = \left(\frac{\varepsilon}{R}\right)^{\frac{1}{a}}$, then for all

$$
\| (w(z_1), h(z_1)) - (w(z_2), h(z_2)) \| \leq R|z_1 - z_2|^a < \varepsilon
$$

Whenever $|z_1 - z_2| < \delta$; whence it follows that $S_a(\theta; R)$ is uniformly bounded and its elements are continuous of the same order. Therefore, by Arzela-Ascoli Theorem, the ball $S_a(\theta; R)$ is a compact subset of $\left(\bigcap_{a,2}^{a,2}(\bar{G})\right)$.

We may now apply Theorem 1 (notice that (A2) cannot occur, because for any solution $(w, h) \in C^{a,2}(\bar{G})$ of the system

$$
\begin{align*}
   w(z) &= A_f(z, w(z), h(z), T_g g_1 (\cdot, w(\cdot), h(\cdot))(z)) \\
   h(z) &= A'_f(z, w(z), h(z), T_g g_2 (\cdot, w(\cdot), h(\cdot))(z))
\end{align*}
$$

for each $\lambda \in (0,1)$ we have $\|(w, h)\|_{a,2} \neq R$, that is, for any solution $(w, h) \in C^{a,2}(\bar{G})$ of the equation $(w, h) = \lambda A(w, h)$ for each $\lambda \in (0,1)$ we have $(w, h) \in S_a[\theta; R]$ to deduce that $A$ has a fixed point in $\bar{U}$, or equivalently, (1.2) has a solution in $S_a(\theta; R)$. This completes the proof.

**References**


